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The Gravitational Reaction Force on a Particle in the Schwarzschild Background

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We formulate a new method to calculate the gravitational reaction force on a particle of mass μ orbiting a massive black hole of mass M. In this formalism, the tail part of the retarded Green function, which is responsible for the reaction force, is calculated at the level of the Teukolsky equation. Our method naturally allows a systematic post-Minkowskian (PM) expansion of the tail part at short distances. As a first step, we consider the case of a Schwarzschild black hole and explicitly calculate the first post-Newtonian (1PN) tail part of the Green function. There are, however, a couple of issues to be resolved before explicitly evaluating the reaction force by applying the present method. We discuss possible resolutions of these issues.

I. INTRODUCTION

In the black hole perturbation approach to gravitational radiation from an inspiralling compact binary, one calculates the gravitational waves emitted by a point particle orbiting a black hole, assuming the mass of the particle (μ) is much less than the mass of the black hole (M). At the lowest order in μ/M, the particle moves along a geodesic trajectory in the black hole geometry. Already at this lowest order, this approach has been proved to be very powerful for evaluating general relativistic corrections to the gravitational wave forms from a compact binary, even for neutron star-neutron star (NS-NS) binaries [1]. However, in such calculations one inevitably has to assume that the radiation reaction time scale is sufficiently long compared to the orbital period, so that the adiabatic approximation to the orbital evolution is valid. That is, the orbital evolution is determined by the energy and angular momentum balance between the orbit and the graviational radiation.

However, there are several situations in which such an approximation cannot be justified. For example, when the particle is nearly in the last stable circular orbit, the radiation reaction time scale becomes comparable to the orbital period. When the particle is in a very eccentric orbit or in a scattering orbit, presumably one has to determine the orbital evolution not by averaging over many periods but by each scattering event.

Another case is when the massive object is in fact a rotating black hole. In this case the geometry is Kerr, and the geodesic orbit is determined not only by the energy and the z-component of the angular momentum but also by the Carter constant [2]. Since the Carter constant is not associated with the Killing vector field of the Kerr geometry, its evolution cannot be determined by evaluating the gravitational radiation at infinity, even though the adiabatic approximation may still be valid.

In order to treat these situations, it is therefore necessary to evaluate the correction to the equation of motion of a particle in the background geometry. That is, we must know the reaction force of $O(\mu/M)$ acting on the particle,

$$\frac{D^2 z^\mu(\tau)}{d\tau^2} = F^\mu(z)\,. \tag{1.1}$$

Formally F^μ is given by [3,4]

$$F^\mu = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)\left(2h^{\rm tail}_{\nu\beta;\alpha} - h^{\rm tail}_{\alpha\beta;\nu}\right) u^\alpha u^\beta\,, \tag{1.2}$$

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where $u^\mu = dz^\mu/d\tau$ and $h^{\rm tail}_{\mu\nu}$ is the tail part of the linear perturbation caused by the particle, i.e., the contribution from a part of the retarded Green function that has support inside the past lightcone. Thus our task is to evaluate the tail part of the Green function[‡].

However, since the tail part depends non-locally on the background geometry, it seems practically impossible to evaluate it directly. As a way to avoid this difficulty, Mino proposed a very interesting approach [5]. He noticed that the direct part of the Green function, which has support only on the lightcone, is determined by the local geometry. Hence the corresponding metric perturbation can be evaluated solely with knowledge of the local geometry. On the other hand, in the black hole background, the linear perturbation is known to be described by the Teukolsky formalism [6], and there exists a method to obtain the general solution in a systematic manner [7]. Then, using the transformation obtained by Chrzanowski [8], one can construct the corresponding metric perturbation. Hence, on the one hand, one may construct the full metric perturbation by solving the Teukolsky equation, and on the other hand, one can calculate the direct part of the metric perturbation by analyzing the local geometry. One can then obtain the tail part by subtracting the direct part from the full metric perturbation.

This approach was further advanced by Mino and Nakano [9], and they succeeded in obtaining some interesting results. Yet, their method seems rather cumbersome. It involves many steps of lengthy calculations, mainly resulting from the different choice of gauges in which the full metric perturbation and its direct part are evaluated.

In this paper, we propose a new, possibly more practical method to evaluate the tail part of the perturbation. As noted by Mino [5], what one needs to do is to extract the direct part of the Green function. But this does not have to be done on the level of the metric perturbation. Instead, if one considers the Teukolsky equation in the spacetime domain, one can evaluate the direct part of the Green function for the Teukolsky equation directly, without transforming it to the metric perturbation. By Fourier-harmonic expanding the direct part obtained in this manner, one can then systematically evaluate the tail part. Then the corresponding metric perturbation and the reaction force F^μ can be calculated by using the Chrzanowski transformation [8].

A technical problem in this approach is that the Teukolsky equation has a very non-trivial angular spin weight dependence, even in the limit of flat spacetime, $M \to 0$. Thus the evaluation of the direct part becomes very involved. As a way to avoid this problem, we consider the transformation of the Teukolsky equation with spin weight s to a Klein-Gordon-type equation (i.e., with spin weight $s = 0$). In the case of the Schwarzschild background, this is essentially the Chandrasekhar transformation that transforms the Teukolsky equation into the Regge-Wheeler equation [10]. In the case of the Kerr background, there is no known transformation of this kind. One possibility is to use the transformation given by Sasaki and Nakamura [11] in a perturbative manner, with respect to the spin parameter a/M of the Kerr geometry. In this paper, however, we focus on the case of the Schwarzschild black hole background and defer the extension to the Kerr case to a future work.

There are also a couple of issues to be resolved before applying the present method to the actual calculation of the reaction force. First, the Teukolsky equation, by its nature, does not give the $\ell = 0$ and 1 spherical harmonic components of the perturbation. Hence the tail part of these components must be calculated separately. Second, and most importantly, since the reaction force F^μ is a gauge-dependent notion, its physical effect can be clarified only after we obtain the $O(\mu)$ correction to the gravitational radiation emitted to infinity, but what we have at hand is only the linear-order metric perturbation. Thus, strictly speaking, there is no point in evaluating the reaction force unless we have a method to evaluate the corresponding correction to the gravitational radiation.

There have been other proposals for the regularization of the self force. Barack and Ori [12] proposed a mode-sum regularization scheme. The calculation of the reaction force in their approach has two steps. First, each Fourier-harmonic mode of the bare self force is calculated. Then the sum over all modes is made subject to certain regularization procedure, which requires several regularization parameters. The calculation of the bare force on a scalar charge is done in [13–15]. The analytical determination of the regularization parameters is carried out using a local perturbative analysis in [12,16]. A method similar to that in [12] was proposed independently by Lousto [17]. Very recently, using the mode-sum regularization prescription, the regularization of the scalar and electromagnetic self force in the spacetime of spherical shells has been discussed by Burko, Liu and Soen [18].

This paper is organized as follows. In Sec. II, we consider the Chandrasekhar transformation in the spacetime domain and discuss the nature of the direct part and the tail part of the Regge-Wheeler Green function. Then we give the Fourier-harmonic expanded form of the Teukolsky and Regge-Wheeler Green functions. In Sec. III, we formulate a method to calculate the tail part of the Regge-Wheeler Green function. Then, the tail part correct to the first post-Minkowskian (1PM) order is given under the short distance approximation, and an explicit expression of the tail in the first post-Newtonian (1PN) order is given. Then in Sec. IV, the procedure to obtain the regularized reaction

[‡]In this paper, we refer to the 'retarded Green function' simply as the 'Green function'.

force is discussed. Finally, in Sec. V, we summarize the results and discuss a possible way to resolve the remaining issues mentioned above.

II. REGGE-WHEELER AND TEUKOLSKY GREEN FUNCTIONS

A. Chandrasekhar transformation in spacetime domain

The Teukolsky equation in the spacetime domain is expressed as

$$ {}_s\Box\, {}_s\Psi \equiv \left[\nabla^\mu\nabla_\mu + \frac{2s}{r^2}\left\{\left(-r+\frac{Mr^2}{\Delta_K}\right)\partial_t + (r-M)\partial_r + i\frac{\cos\theta}{\sin^2\theta}\partial_\varphi + \frac{1-s\cot^2\theta}{2}\right\}\right]{}_s\Psi = -4\pi\, {}_s\mathcal{T}\,, \tag{2.1}$$

where $s=0$, ±1 or ±2, and $\Delta_K = r(r-2M)$. We note that ${}_s\Psi$ has spin weight s with respect to the spatial rotation of the tetrad, but it is a spacetime scalar. The quantity ${}_s\mathcal{T}$ is the source term whose explicit form can be found in [6]. Although our interest is in the gravitational perturbation, and it is described by the Teukolsky equation with $|s|=2$, we leave the value of s unfixed until we need to fix it. As we see below, this makes it easy to identify the tail part of the Green function.

The difficulty of directly dealing with the Teukolsky equation comes from the non-trivial angular dependence in its zeroth and first derivative terms. In particular, this causes non-conventional behavior of the Green function even in the flat spacetime limit. To overcome this difficulty, we consider a transformation of the Teukolsky equation. Since there is a one-to-one correspondence between the Teukolsky equation with spin weights s and $-s$ (the Starobinsky-Teukolsky identities), we concentrate on negative spin weights $s=0$, -1 and -2. For $s\le 0$, we introduce a new field variable ${}_sX$ by setting

$$ {}_s\Psi = \bar{\eth}^{|s|}\left(\frac{\Delta_K}{r}\right)^{|s|}\left(\partial_r - \frac{r^2}{\Delta_K}\partial_t\right)^{|s|} r^{|s|}\, {}_sX\,, \tag{2.2}$$

where

$$ \begin{aligned} \eth &= -\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi - s\cot\theta\right),\\ \bar{\eth} &= -\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi + s\cot\theta\right). \end{aligned} \tag{2.3}$$

It should be noted that $\eth$ ($\bar{\eth}$) operating on a quantity of spin weight s yields a quantity of spin weight $s+1$ $(s-1)$. In particular, we have

$$ \begin{aligned} \eth\, ({}_sY_{\ell m}) &= [\ell(\ell+1) - s(s+1)]^{1/2}\, {}_{s+1}Y_{\ell m}\,,\\ \bar{\eth}\, ({}_sY_{\ell m}) &= -\,[\ell(\ell+1) - s(s-1)]^{1/2}\, {}_{s-1}Y_{\ell m}\,,\\ \bar{\eth}\eth\, ({}_sY_{\ell m}) &= -\,[\ell(\ell+1) - s(s+1)]\, {}_sY_{\ell m}\,, \end{aligned} \tag{2.4}$$

where the ${}_sY_{\ell m}$ are the normalized spherical harmonics of spin weight s. This implies that ${}_sX$ has spin weight $s+|s|=0$ for $s\le 0$. If expanded in terms of $e^{-i\omega t}$ and ${}_sY_{\ell m}$, the radial part of the above transformation is just the Chandrasekhar transformation that relates homogeneous Teukolsky functions to Regge-Wheeler functions, apart from the overall normalization constant that depends on ℓ and ω. A conceptually important difference is that here we do not associate ${}_sX$ with the spin weight s, but with $s=0$.

Inserting (2.2) into (2.1), we find

$$ \begin{aligned} {}_s\Box\, {}_s\Psi &= \bar{\eth}^{|s|}\left(\frac{\Delta_K}{r}\right)^{|s|}\frac{1}{r^2}\left(\partial_r - \frac{r^2}{\Delta_K}\partial_t\right)^{|s|} r^{|s|+2}\left[\nabla^\mu\nabla_\mu + s^2\,\frac{2M}{r^3}\right]{}_sX = -4\pi\, {}_s\mathcal{T}\,,\\ \Rightarrow&\quad \left[\nabla^\mu\nabla_\mu + s^2\,\frac{2M}{r^3}\right]{}_sX = -4\pi\, {}_s\tilde{\mathcal{T}}\,. \end{aligned} \tag{2.5}$$

We see that the equation for ${}_sX$ has the standard Klein-Gordon type structure with a potential term that vanishes sufficiently rapidly at infinity irrespective of the spin weight. Note that its radial part is just the Regge-Wheeler

equation. It then follows that ${}_sX$ can be naturally expanded in terms of the scalar (i.e., $s=0$) spherical harmonics even for nonzero s, in accordance with the discussion above. A very important point to be noted is the following. If one constructs the Green function of Eq. (2.5) for ${}_sX$, its direct part is exactly the same for all the spin weights. This is because the local causal structure of a hyperbolic equation depends only on the derivative terms of the equation (i.e., the $\nabla^\mu\nabla_\mu$ part). This implies that the part that should be discarded from the Green function is independent of s if done for the variable ${}_sX$. In other words, any s-dependent terms in the Green function for ${}_sX$ belong to the tail part.

B. Green functions for ${}_s\Psi$ and ${}_sX$

We first consider the Green function ${}_sG^{\rm ret}$ for Eq. (2.5). It satisfies

$$\left[\nabla^\mu\nabla_\mu + s^2\,\frac{2M}{r^3}\right]\,{}_sG^{\rm ret}(x,x') = -\frac{\delta(t-t')\delta(r-r')\delta^2(\Omega-\Omega')}{rr'}\,. \tag{2.6}$$

Its Fourier-harmonic expansion is expressed as

$$\begin{aligned}{}_sG^{\rm ret}(x,x')&= \int\frac{d\omega}{2\pi}\sum_{\ell m}\,{}_sG_{\ell m\omega}(r,r')\,{}_0Y_{\ell m}(\Omega)\,\overline{{}_0Y_{\ell m}(\Omega')}\,e^{-i\omega(t-t')}\,;\\ {}_sG_{\ell m\omega}(r,r') &= -\frac{\omega}{|\omega|}\,\frac{1}{W(X_\ell^{\rm up},X_\ell^{\rm in})}\,\frac{X_\ell^{\rm up}(r_>)\,X_\ell^{\rm in}(r_<)}{rr'}\,,\end{aligned} \tag{2.7}$$

where $r_> = \max\{r,r'\}$, $r_< = \min\{r,r'\}$, ${}_0Y_{\ell m}$ is the usual (spin weight 0) spherical harmonic function, $W(X_\ell^{\rm up},X_\ell^{\rm in})$ is the Wronskian,

$$W(X_\ell^{\rm up},X_\ell^{\rm in}) = \left(\frac{d}{dr^*}X_\ell^{\rm up}\right)X_\ell^{\rm in} - X_\ell^{\rm up}\frac{d}{dr^*}X_\ell^{\rm in}\,, \tag{2.8}$$

and $X_\ell^{\rm up}$ and $X_\ell^{\rm in}$ are the homogeneous solutions of the Regge-Wheeler equation having the asymptotic behaviors,[§]

$$\begin{aligned}X_\ell^{\rm in} &\sim \begin{cases}\mathcal{A}^{\rm in}_{\ell\omega}e^{-i\omega r^*} + \mathcal{A}^{\rm out}_{\ell\omega}e^{i\omega r^*} & (r\to\infty),\\ e^{-i\omega r^*} & (r\to 2M),\end{cases}\\ X_\ell^{\rm up} &\sim \begin{cases}e^{i\omega r^*} & (r\to\infty),\\ \mathcal{D}^{\rm in}_{\ell\omega}e^{-i\omega r^*} + \mathcal{D}^{\rm out}_{\ell\omega}e^{i\omega r^*} & (r\to 2M).\end{cases}\end{aligned} \tag{2.9}$$

It then follows that $W = 2i\omega\mathcal{A}^{\rm in}_{\ell\omega} = 2i\omega\mathcal{D}^{\rm out}_{\ell\omega}$.

To 1PM order, the necessary formulas for the radial functions $X_\ell^{\rm in}$ and $X_\ell^{\rm up}$ may be found in Poisson and Sasaki [20] and Leonard and Poisson [21]. (Their explicit expressions are given in Appendix A.) For $\omega>0$, they take the form

$$X_\ell^{\rm in}(z) = z\left[j_\ell(z) + \epsilon\,\xi_\ell^{\rm in}(z) + O(\epsilon^2)\right], \tag{2.10}$$

$$X_\ell^{\rm up}(z) = z\left[h_\ell^{(1)}(z) + \epsilon\,\xi_\ell^{\rm up}(z) + O(\epsilon^2)\right], \tag{2.11}$$

where $\epsilon = 2M\omega$, $z=\omega r$, and the functions $\xi_\ell^{\rm in}$ and $\xi_\ell^{\rm up}$ are defined in Eqs. (A1) and (A2). As noted in Appendix A, for $\omega<0$ (i.e, for $z<0$), they should be replaced by $\overline{X_\ell^{\rm in}(|z|)}$ and $\overline{X_\ell^{\rm up}(|z|)}$, respectively. With this understanding, we find

$${}_sG_{\ell m\omega}(r,r') = i\omega\,\left[j_\ell(z_<)h_\ell^{(1)}(z_>) + \epsilon\,\left(j_\ell(z_<)\xi_\ell^{\rm up}(z_>) + \xi_\ell^{\rm in}(z_<)h_\ell^{(1)}(z_>)\right) + O(\epsilon^2)\right]. \tag{2.12}$$

Hence

[§] In this paper, we follow the notation of Chrzanowski and Misner [19] for the homogeneous solutions.

$$\sum_m {}_sG_{\ell m\omega} Y_{\ell m}(\Omega)\overline{Y_{\ell m}(\Omega')}$$

$$= \frac{i\omega}{4\pi}\,(2\ell+1)\left[j_\ell(z_<)h^{(1)}_\ell(z_>) + \epsilon\,\left(j_\ell(z_<)\xi^{\rm up}_\ell(z_>) + \xi^{\rm in}_\ell(z_<)h^{(1)}_\ell(z_>)\right) + O(\epsilon^2)\right] P_\ell(\mu)\,, \tag{2.13}$$

where $\mu = \mathbf{\Omega}\cdot\mathbf{\Omega}'$.

Now, given the Green function for ${}_sX$, the Green function for ${}_s\Psi$ is given as follows. The Teukolsky functions ${}_sR^{\rm in}_\ell$, ${}_sR^{\rm up}_\ell$ and the Regge-Wheeler functions ${}_sX^{\rm in}_\ell$, ${}_sX^{\rm up}_\ell$ are related by the Chandrasekhar transformation,

$${}_sR^{\rm in}_\ell = {}_s\chi^{\rm in}_\ell\,{}_sC_\omega\,{}_sX^{\rm in}_\ell\,, \quad {}_sR^{\rm up}_\ell = {}_s\chi^{\rm up}_\ell\,{}_sC_\omega\,{}_sX^{\rm up}_\ell\,, \tag{2.14}$$

where ${}_sC_\omega$ $(s = 0, \pm1, \pm2)$ are the Chandrasekhar operators [10,20,21] and ${}_s\chi^{\rm in}_\ell$ and ${}_s\chi^{\rm up}_\ell$ are certain normalization coefficients (the case of $s = 0$ is trivial, since ${}_0R = {}_0X$). For $s = -2$, the operator ${}_{-2}C_\omega$ is defined by

$$C_\omega = \omega r^2 f\,\mathcal{L}\,f^{-1}\mathcal{L}\,r\,, \tag{2.15}$$

where

$$\mathcal{L} = f\frac{d}{dr} + i\,\omega\,, \quad f = \left(1 - \frac{2M}{r}\right), \tag{2.16}$$

and we have omitted the spin index of ${}_{-2}C_\omega$ for simplicity and denoted it simply by C_ω, since we are interested only in the case $s = -2$. The coefficients ${}_{-2}\chi^{\rm in}_\ell$ and ${}_{-2}\chi^{\rm up}_\ell$ are given by

$$\chi^{\rm in}_\ell = \frac{16\,(1-2\,i\,M\omega)(1-4\,i\,M\omega)(M\omega)^3}{(\ell-1)\ell(\ell+1)(\ell+2) - 12\,i\,M\omega}\,, \quad \chi^{\rm up}_\ell = -\frac{1}{4}\,, \tag{2.17}$$

where we have also omitted the spin indices for simplicity. In addition, in the rest of the paper, we denote ${}_{-2}\Psi$ by ψ_4, in accordance with the conventional notation for the Weyl scalar.

The Green function for the Teukolsky equation is constructed from the above homogeneous functions as given by Eq. (4.1) below. Except for the Wronskian $W(R^{\rm up}_\ell, R^{\rm in}_\ell)$, however, the explicit form of the Green function for the Teukolsky equation is not needed for the following calculation, as will be clarified later.

III. TAIL PART AT SHORT DISTANCES

As usual, we write the Green function in the Hadamard form,

$${}_sG^{\rm ret}(x,x') = \theta(\Sigma(x),x')\left(\frac{u}{4\pi}\delta(\sigma) + \theta(-\sigma)\,{}_sv\right), \tag{3.1}$$

where σ is half the squared geodetic distance between x and x', u is a bi-scalar that depends only on the local geometry, and ${}_sv$ is a bi-scalar that describes the tail effects. We have a unique solution for each of these bi-scalars. As noted in the previous section, the direct part of ${}_sG^{\rm ret}$ is independent of s, and the bi-scalar u is given simply by that in the case of the scalar d'Alembertian [22],

$$u = \left(\frac{-\det|\sigma_{;\alpha;\beta'}|}{\sqrt{g(x)g(x')}}\right)^{1/2} \tag{3.2}$$

Let us denote the direct part of ${}_sG^{\rm ret}$ by $G^{\rm dir}$:

$$G^{\rm dir}(x,x') = \theta(\Sigma(x),x')\,\frac{u}{4\pi}\,\delta(\sigma). \tag{3.3}$$

Our task is to evaluate $G^{\rm dir}$ in the Fourier-harmonic expanded form at short distances.

Since the full evaluation of the direct part is difficult, we consider its post-Minkowskian (PM) expansion. To do so, it is more convenient to work in the isotropic coordinates than in the Schwarzschild coordinates. The radial coordinates of these two coordinates are related as $r = r_I(1 + M/2r_I)^2$. Here r denotes the conventional Schwarzschild radial coordinate and r_I denotes the isotropic radial coordinate. It should be noted that the isotropic coordinates satisfy

the harmonic gauge condition to 1PM order. The bi-scalar $\sigma(x,x')$ accurate to 1PM order can be found in a classic paper by Thorne and Kovacs [23], or in Leonard and Poisson [21].

We will come back to the explicit evaluation of the 1PM tail part later. Here, we first describe the formal procedure to obtain the tail part. Let us express the Fourier-harmonic components of the full Regge-Wheeler Green function (2.7) as

$$
{}_sG(\ell m\omega;\, x,x') = {}_sG_{\ell m\omega}(r,r')\, {}_0Y_{\ell m}(\Omega)\overline{{}_0Y_{\ell m}(\Omega')}\, e^{-i\omega(t-t')}\,. \tag{3.4}
$$

Correspondingly, the Fourier-harmonic components of the direct part, which is s-independent, may be expressed as

$$
G^{\rm dir}(\ell m\omega;\, x,x') = G^{\rm dir}_{\ell m\omega}(r,r')\, {}_0Y_{\ell m}(\Omega)\overline{{}_0Y_{\ell m}(\Omega')}\, e^{-i\omega(t-t')}\,. \tag{3.5}
$$

The tail part of the Green function, i.e., the regularized Green function, is obtained by subtracting the direct part from the full Green function:

$$
\begin{aligned}
{}_sG^{\rm tail}(\ell m\omega;\, x,x') &= {}_sG^{\rm tail}_{\ell m\omega}(r,r')\, {}_0Y_{\ell m}(\Omega)\overline{{}_0Y_{\ell m}(\Omega')}\, e^{-i\omega(t-t')}\,; \\
{}_sG^{\rm tail}_{\ell m\omega}(r,r') &= {}_sG_{\ell m\omega}(r,r') - G^{\rm dir}_{\ell m\omega}(r,r')\,.
\end{aligned} \tag{3.6}
$$

Since this subtraction is done on a "mode-by-mode" basis, we call our procedure the "mode-by-mode regularization" of the Green function.

Now let us return to the evaluation of the first post-Minkowskian (1PM) order tail. To 1PM order, the geodesic connecting x and x' can be approximated by a straight line on the flat background [23,21]:

$$
\xi^\mu(\lambda) = x'^\mu + \lambda X^\mu\,; \quad X^\mu = x^\mu - x'^\mu\,. \tag{3.7}
$$

Here the affine parameter λ is normalized so that $\xi^\mu(0) = x'^\mu$ and $\xi^\mu(1) = x^\mu$. We then have

$$
\begin{aligned}
\sigma(x,x') &= \frac{1}{2}\left(\eta_{\mu\nu} + \int_0^1 h_{\mu\nu}(\xi(\lambda))d\lambda\right) X^\mu X^\nu \\
&= \frac{1}{2}\left(-\Delta t^2(1-F) + R^2(1+F)\right)\,; \qquad F = \int_0^1 \frac{2M}{r_I(\lambda)}d\lambda\,,
\end{aligned} \tag{3.8}
$$

where $\Delta t = t - t'$ and $R^2 = \delta_{ij}(x^i - x'^i)(x^j - x'^j)$. The 1PM correction factor F can be calculated exactly as

$$
\begin{aligned}
F &= \frac{2M}{R}\ln\frac{(r_I + r_I' + R)}{(r_I + r_I' - R)} = \frac{2M}{r_I + r_I'}\sum_{n=0}^{\infty}\frac{2}{2n+1}\left(\frac{R}{r_I + r_I'}\right)^{2n} \\
&= \frac{4M}{r_I + r_I'} + \frac{4M}{3(r_I + r_I')}\left(\frac{R}{r_I + r_I'}\right)^2 + O(R^4),
\end{aligned} \tag{3.9}
$$

where the short-distance approximation is applied to the second line. The bi-scalar u accurate to 1PM order is unity, because of the Ricci flatness of the Schwarzschild geometry. Thus the direct part of the Green function becomes

$$
\begin{aligned}
G^{\rm dir}(x,x') &= \frac{\theta(\Delta t)}{4\pi}\, 2\delta\left(-\Delta t^2(1-F) + R^2(1+F)\right) \\
&= \frac{\delta\left(\Delta t - R(1+F)\right)}{4\pi R}\,.
\end{aligned} \tag{3.10}
$$

To subtract the direct part from the Green function, we take the Fourier-harmonic expansion of the direct part. The Fourier transformation of $G^{\rm dir}$ is

$$
\begin{aligned}
\tilde{G}^{\rm dir}_\omega(\boldsymbol{x},\boldsymbol{x'}) &= \int G^{\rm dir} e^{i\omega\Delta t} d\Delta t = \frac{e^{i\omega R(1+F)}}{4\pi R} \\
&= \frac{e^{i\omega R}}{4\pi R}\left(1 + i\omega RF + O(M^2)\right)\,.
\end{aligned} \tag{3.11}
$$

Using the relation

$$
\frac{e^{i\omega R}}{R} = \sum_{\ell=0}^{\infty}(2\,\ell+1)\, j_\ell(\omega r_{I<})\, h^{(1)}_\ell(\omega r_{I>})\, P_\ell(\mu)\,, \tag{3.12}
$$

and re-expressing r_I in terms of the Schwarzschild radial coordinate r, we find

$$\tilde{G}^{\rm dir}_{\omega}(\boldsymbol{x},\boldsymbol{x'})= \sum_{\ell=0}^{\infty}\frac{i\omega}{4\pi}\,(2\ell+1)\Bigg[j_\ell(z_<)h^{(1)}_\ell(z_>)+\frac{\epsilon}{2}\Bigg(\left(\frac{1}{z_<}+\frac{1}{z_>}\right)j_\ell(z_<)\,h^{(1)}_\ell(z_>) + \frac{z_<}{z_>}\,j'_\ell(z_<)\,h^{(1)}_\ell(z_>)+\frac{z_>}{z_<}\,j_\ell(z_<)\,{h^{(1)}}'_\ell(z_>)\Bigg)+O\left(\epsilon^2,|z-z'|^3\right)\Bigg]P_\ell(\mu)\,, \tag{3.13}$$

where $z=\omega r$, $z'=\omega r'$ and $\epsilon=2M\omega$, as before. Thus the 1PM direct part of the radial Green function is found as

$$G^{\rm dir}_{\ell m\omega}(r,r')= i\omega\Bigg[j_\ell(z_<)h^{(1)}_\ell(z_>)+\frac{\epsilon}{2}\Bigg(\left(\frac{1}{z_<}+\frac{1}{z_>}\right)j_\ell(z_<)\,h^{(1)}_\ell(z_>) + \frac{z_<}{z_>}\,j'_\ell(z_<)\,h^{(1)}_\ell(z_>)+\frac{z_>}{z_<}\,j_\ell(z_<)\,{h^{(1)}}'_\ell(z_>)\Bigg)+O\left(\epsilon^2,|z-z'|^3\right)\Bigg]\,. \tag{3.14}$$

The 1PM tail is obtained by subtracting the above from the full 1PM Green function, (2.12).

Further performing the post-Newtonian expansion, the 1PN direct Green function is obtained explicitly as

$$G^{\rm dir}_{\ell m\omega}(r,r') = \frac{1}{2\ell+1}\,\frac{1}{r'}\left(\frac{r}{r'}\right)^{\ell} \times\left(1+\frac{M\,(\ell+1)}{r'}+\frac{1}{2}\,\frac{\omega^2 r'^2}{2\,\ell-1}-\frac{M\,\ell}{r}-\frac{1}{2}\,\frac{\omega^2 r^2}{2\,\ell+3}+O(|r-r'|^3)\right)\,. \tag{3.15}$$

Expanding the full 1PM Green function (2.12) to 1PN order, and subtracting the above direct part (3.15) from the result, we find

$${}_sG^{\rm tail}_{\ell m\omega}(r,r') = \frac{s^2}{2\ell+1}\,\frac{1}{r'}\left(\frac{r}{r'}\right)^{\ell}\left(-\frac{M}{(\ell+1)\,r'}+\frac{M}{\ell\,r}\right)\,. \tag{3.16}$$

It is worthwhile to note that the 1PN tail comes solely from the s-dependent part of the Regge-Wheeler equation. Thus the scalar d'Alenbertian $\nabla^\mu\nabla_\mu$ contains no 1PN tail, but only the correction to the light cone. In particular, this implies the 1PN tail is absent for a particle with scalar charge, in accordance with the recent result of Burko, Liu and Soen [18].

IV. REGULARIZATION

In this section, we give the procedure to calculate the regularized Weyl scalar, the tail part of the metric perturbation and the gravitational reaction force at the position of a point particle.

A. Tail part of the metric perturbation

From the homogeneous solutions $X^{\rm in}_\ell$ and $X^{\rm up}_\ell$ of the Regge-Wheeler equation, we obtain the corresponding solutions of the Teukolsky equation by the Chandrasekhar transformation (2.14). Using the method described in the previous section, the radial part of the regularized Green function for the Weyl scalar ${}_s\mathcal{G}_{lm\omega}(r,r')$ is found as

$${}_s\mathcal{G}_{\ell m\omega}(r,r') = -2\,\pi\,\frac{W(X^{\rm up}_\ell,X^{\rm in}_\ell)}{W(R^{\rm up}_\ell,R^{\rm in}_\ell)}\chi^{\rm in}_\ell\chi^{\rm up}_\ell\,C_\omega(r)\,C_\omega(r')\,{}_sG^{\rm tail}_{\ell m\omega}(r,r')\,, \tag{4.1}$$

where $s=-2$ and $C_\omega(r)$ is the Chandrasekhar operator defined by Eq. (2.15). The factor -2π is inserted in the above formula for ${}_s\mathcal{G}_{\ell m\omega}$, so that the Weyl scalar ψ_4 is given in the form

$$r^4\psi_4(x) = \int\frac{d\omega}{2\pi}\sum_{\ell m}\left[\int dr'\,{}_{-2}\mathcal{G}_{\ell m\omega}(r,r')\,T_{\omega\ell m}(r')r'^{-2}(r'-2M)^{-2}\right]{}_{-2}Y_{\ell m}(\Omega)e^{-i\omega t}\,, \tag{4.2}$$

with the source term $T_{\omega\ell m}$ defined in Appendix D, in accordance with the definition of Poisson and Sasaki [20].

Once we have the regularized Green function for the Weyl scalar, the regularized metric perturbation can be constructed by the Chrzanowski transformation [8]. In the present case, we calculate the tail part (i.e., the regularized part) of the metric perturbation as follows. We decompose the radial part of the regularized Green function for the metric perturbation into parts having different spin weights as

$$G^{\ell m\omega}_{(s,s')}(r,r') = \frac{32\,{}_sp_\ell\,{}_{s'}p_\ell}{({}_0p_\ell)^2}\,{}_sd_\omega(r)\,{}_{s'}D^\dagger_\omega(r')\,{}_{-2}\mathcal{G}_{\ell m\omega}(r,r')\,, \tag{4.3}$$

where the coefficients ${}_sp_\ell$, and the operators ${}_sd_\omega$ and ${}_{s'}D^\dagger_\omega$ are defined in Eqs. (B13), (B14) and (B15), respectively, of Appendix B. Then the tail part of the metric perturbation is expressed as

$$\begin{aligned} h_{\mu\nu}(\ell m\omega;x) = \Big\{ & l_\mu l_\nu \left(\sum_s \int_{2M}^\infty dr' r'^{-2}(r'-2M)^{-2} G^{\ell m\omega}_{(0,s)}(r,r')\,{}_s\overline{T}_{\omega\ell m}(r')\right) {}_0Y_{\ell m}(\theta,\phi)e^{-i\omega t} \\ & -(l_\mu \bar{m}_\nu + \bar{m}_\mu l_\nu)\left(\sum_s \int_{2M}^\infty dr' r'^{-2}(r'-2M)^{-2} G^{\ell m\omega}_{(1,s)}(r,r')\,{}_s\overline{T}_{\omega\ell m}(r')\right) {}_1Y_{\ell m}(\theta,\phi)e^{-i\omega t} \\ & +\bar{m}_\mu \bar{m}_\nu \left(\sum_s \int_{2M}^\infty dr' r'^{-2}(r'-2M)^{-2} G^{\ell m\omega}_{(2,s)}(r,r')\,{}_s\overline{T}_{\omega\ell m}(r')\right) {}_2Y_{\ell m}(\theta,\phi)e^{-i\omega t}\Big\}\,, \end{aligned} \tag{4.4}$$

where $\{l_\alpha\} = \{-1, r/(r-2M), 0, 0\}$ and $\{\bar{m}_\alpha\} = \{0, 0, 1, -i\sin\theta\}r/\sqrt{2}$ are the Kinnersley null tetrad, and ${}_s\overline{T}_{\omega\ell m}$ is the complex conjugate of ${}_sT_{\omega\ell m}$. We note that the above metric perturbation is given under the ingoing radiation gauge condition, defined by

$$h_{\mu\nu}\,l^\mu = 0\,, \quad h_\mu{}^\mu = 0\,. \tag{4.5}$$

It should be noted that the Chandrasekhar transformation and the subsequent Chrzanowski transformation in their original forms can be applied only in the formal sense in the above procedure. In practice, one should modify those transformations as discussed in Appendix C, in order to subtract out the direct part induced by application of the original transformations. Let us explain the reason. The Chandrasekhar transformation involves the second order derivatives. If one takes the second-order derivatives of the Green function for the Regge-Wheeler equation in the spacetime domain, the step function $\theta(-\sigma)$ in front of the tail part is transmuted to the direct part of the Green function for the Weyl scalar. This phenomenon shows up in the Fourier-harmonic domain, as demonstrated in Appendix C. Thus one should modify the Chandrasekhar transformation so as to subtract this induced direct part. As the Chandrasekhar transformation, the Chrzanowski transformation also involves the second-order radial derivatives. Hence, it is necessary to introduce a modified Chrzanowski transformation to subtract the induced direct part. In the actual calculation of the metric perturbation (or the reaction force), however, we do not introduce the modified Chandrasekhar and Chrzanowski transformations individually, but construct operators that contain only the first-order derivatives and that directly transform the Green function for the Regge-Wheeler variable to that for the components of the metric perturbation (or the reaction force) as illustrated at the end of Appendix C. The explicit expressions for the transformation operators are given in Appendix B. We note that thus constructed operators are same as the original operators with higher derivatives when applied to the full Green function.

When we take the limit in which the position coincides with the particle, the following facts should be kept in mind. Let us denote the position of a particle by $\{z^\alpha\} = \{t_0, r_0, \theta_0, \phi_0\}$ and a field point by $\{x^\mu\} = \{t, r, \theta, \phi\}$. Because of the causal nature of the tail part, strictly speaking, the coincidence limit has to be taken by first taking the spatial limit as $r \to r_0$, and then taking the limit $t \to t_0$. On the other hand, the coincidence limit for the angular coordinates may be taken at any time, as the metric perturbation depends on the angular coordinates only through the regular functions ${}_sY_{\ell m}$. But as long as we employ the post-Newtonian (PN) expansion, we may take any order of the coincidence limit, because the light cone flattens out in the PN expansion. Therefore, when we evaluate a quantity in the coincidence limit, we take the limit $t \to t_0$ first, followed by the spatial coincidence limit within the context of the PN expansion. After taking the limit, we perform the summation over ℓ, m and ω.

Here it is worthwhile to note the following. If we take the mode summation over ℓ, m and ω before we take the spatial coincidence limit, the result depends on whether the radial limit is taken from r smaller than r_0 or larger than r_0. This is a reflection of the fact that the first-order radial derivatives remaining in the final operators that transform the Regge-Wheeler variable to the metric perturbation (or to the reaction force) induce a 'quasi-direct' part that has a direction-dependent limit. Nevertheless, the avarage of the two different limits turns out to be the same as the limit obtained by first taking the coincidence limit before the mode summation. This can be explained by considering the first derivative of the step function in the spacetime domain. It has a direction-dependent limit, but this difference disappears when averaged over all directions.

B. Force for a circular orbit

The form of the reaction force which describes the deviation from the geodesic motion on the background is given by Eq. (1.2). Here we focus on circular motion. Since only the t- and ϕ-components of the four velocity are non-zero in this case, the radial component of the reaction force can be expressed as

$$F^r(x) = -\frac{1}{2}g^{rr}\Big[(2h_{rt;t} - h_{tt;r})\,u^t u^t + (2h_{r\phi;\phi} - h_{\phi\phi;r})\,u^\phi u^\phi + 2\,(h_{tr;\phi} + h_{\phi r;t} - h_{t\phi;r})\,u^t u^\phi\Big]_{\rm tail}\,, \tag{4.6}$$

where $[\cdots]_{\rm tail}$ indicates that $h_{\mu\nu;\alpha}$ should be interpreted as the tail part $h^{\rm tail}_{\mu\nu;\alpha}$. Specializing to the case of Schwarzschild background, we have

$$F^r(x) = -\frac{1}{2}g^{rr}\Big[\Big(2h_{rt,t} - h_{tt,r} - \frac{2\,M\,f}{r^2}h_{rr}\Big)\,u^t u^t + \big(2h_{r\phi,\phi} - h_{\phi\phi,r} + 2\,r\sin^2\theta\,f h_{rr}\big)\,u^\phi u^\phi + 2\,(h_{tr,\phi} + h_{\phi r,t} - h_{t\phi,r})\,u^t u^\phi\Big]_{\rm tail}\,. \tag{4.7}$$

Since there is no effect of the gravitational radiation at 1PN order, we may consider only the radial component of the reaction force to that order. Physically, the radial component describes the correction to the radius of the orbit that deviates from the geodesic in the unperturbed background.

As discussed in the previous subsection, we construct the operators that directly trasform the Regge-Wheeler variable into the reaction force expressed in Eq. (4.7). At 1PN order, we find that only the $h^{\rm tail}_{tt,r}$ term contributes. Since this term involves the operation $\partial_{r\,0}\Delta_\omega$, where ${}_0\Delta_\omega$ is defined in Appendix B, Eq. (B19), we reduce the second order derivative in it by using the homogeneous Regge-Wheeler equation, as discussed in the preveous subsection. The result is

$$\begin{aligned}\partial_{r\;0}\Delta_\omega = &\left(-\frac{1}{4}\ell(\ell+1)\left(1-\frac{2M}{r}\right) - i\,\omega r\left(1-\frac{9M}{2r}\right) + \frac{1}{2}\omega^2 r^2 + \frac{M}{r}\left(1-\frac{3M}{r}\right)\right)\frac{1}{r}\partial_r \\ &+\frac{1}{4r^2}\left(-\ell(\ell+1)\left(1+2\,i\,\omega r-\frac{6M}{r}\right) + 2\omega^2 r^2\left(2+i\,\omega r - \frac{9M}{r}\right)\right. \\ &\left.-\frac{4M^2}{r^2}\left(2\,\ell^2+2\,\ell+3-\frac{6M}{r}\right) + 2\,i\,\omega\left(2\ell^2+2\ell+5-\frac{12M}{r}\right)\right)\,.\end{aligned} \tag{4.8}$$

This operator is used to derive the 1PN reaction force term.

After summing over ℓ, m and ω, and taking the coincidence limit, the gravitational reaction force to 1PN order is found to be

$$F^r(z) = \frac{11\,M\mu}{3\,r_0^3}\,. \tag{4.9}$$

We warn that this result by itself has no physical significance. First, it contains contributions only from harmonics with $\ell \geq 2$. We have to calculate the $\ell = 0$ and 1 components to make the force complete. Second, it is obtained in the ingoing radiation gauge, and since we lack a method to calculate the gravitational radiation to second order in μ, the only thing we can do at present is to compare the result with the standard 1PN formula given in the literature. To do so, we have to make a gauge transformation from the ingoing radiation gauge to the harmonic gauge. Only after this procedure can we determine if our result is valid.

V. DISCUSSION

In this paper we have proposed a new method to derive the gravitational reaction force on a point particle in the Schwarzschild background. In this method, the tail part of the metric perturbation, which is responsible for the reaction force, is obtained by first considering the Fourier-harmonic expansion of the Green function for the Regge-Wheeler equation. After calculating the tail part of the Green function, the result is transformed to the metric perturbation through the Chandrasekhar transformation followed by the Chrzanowski transformation. Since

the extraction of the tail part is done at the level of the Green function in the Fourier-harmonic expanded form, we call our method the mode-by-mode regularization of the Green function. An advantage of our method is that it is relatively easy to obtain the tail part of the Green function for the Regge-Wheeler equation if we adopt the post-Minkowskian (or post-Newtonian) expansion.

However, there are still a couple of issues to be resolved. One is that the $\ell = 0$ and 1 components are not included in the metric perturbation obtained with the Chrzanowski transformation. Hence the tail part in them must be derived by some other method. The $\ell = 0$ and 1 components of the full metric perturbation can be solved exactly in the Regge-Wheeler-Zerilli gauge, as shown by Zerilli [24]. Therefore, what we need is a method to identify the tail part in the $\ell = 0$ and 1 components. The other, most crucial issue is that the physical meaning of the resulting reaction force remains unclear, since the reaction force is a gauge-dependent notion. For an ultimate resolution of this essential problem, we need to develop a second-order perturbation theory so as to make it possible to calculate the $O(\mu)$ correction to the gravitational radiation.

Nevertheless, at low post-Newtonian orders, there is a way to resolve these issues. The equations of motion for a two-body system and the gravitational radiation from such a system are investigated extensively in [25–28] in the harmonic gauge. Hence, by performing the gauge transformation to the harmonic gauge, we should be able to clarify the physical effect of our result, as well as to examine the consistency of our result with those of these previous works. As a first step, we are currently working on a consistency check at the 1PN order [29].

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APPENDIX A: THE HOMOGENEOUS 1PM REGGE-WHEELER FUNCTIONS

Here we recapitulate the formulas for the homogeneous 1PM Regge-Wheeler radial functions $X_\ell^{\rm in}$ and $X_\ell^{\rm up}$, which can be found in Poisson and Sasaki [20] and Leonard and Poisson [21].

The formulas are

$$\begin{aligned}
X_\ell^{\rm in}(z) &= 2\mathcal{A}_{\ell\omega}^{\rm in}(-i)^{\ell+1}e^{i\epsilon(\ln(2\epsilon)-\beta_\ell)}(1+\frac{\pi}{2}\,\epsilon)\,z\left[j_\ell(z)+\epsilon\,\xi_\ell^{\rm in}(z)+O(\epsilon^2)\right], \\
&\quad \xi_\ell^{\rm in}(z) = -A_\ell(z)j_\ell(z)+B_\ell(z)n_\ell(z)-\frac{\ell^2-s^2}{2\ell(2\ell+1)}j_{\ell-1}(z)+\frac{(\ell+1)^2-s^2}{2(\ell+1)(2\ell+1)}j_{\ell+1}(z),
\end{aligned} \tag{A1}$$

$$\begin{aligned}
X_\ell^{\rm up}(z) &= i^{\ell+1}e^{-i\epsilon(\ln(2\epsilon)-\beta_\ell)}(1-\frac{\pi}{2}\,\epsilon)\,z\left[h_\ell^{(1)}(z)+\epsilon\,\xi_\ell^{\rm up}+O(\epsilon^2)\right], \\
&\quad \xi_\ell^{\rm up}(z) = (A_\ell(z)-iB_\ell(z))h_\ell^{(1)}(z)+C_\ell(z)j_\ell(z)-\frac{\ell^2-s^2}{2\ell(2\ell+1)}h_{\ell-1}^{(1)}(z)+\frac{(\ell+1)^2-s^2}{2(\ell+1)(2\ell+1)}h_{\ell+1}^{(1)}(z),
\end{aligned} \tag{A2}$$

where $\epsilon = 2M\omega$, $z = \omega r$, and the functions $A_\ell(z)$, $B_\ell(z)$ and $C_\ell(z)$ are given by

$$\begin{aligned}
A_\ell(z) &= \mathrm{Si}(2z)+z^2n_0(z)j_0(z)+\sum_{p=1}^{\ell-1}\left(\frac{1}{p}+\frac{1}{p+1}\right)z^2n_p(z)j_p(z)\,, \\
B_\ell(z) &= \mathrm{Ci}(2z)-\gamma-\ln(2z)+z^2j_0(z)^2+\sum_{p=1}^{\ell-1}\left(\frac{1}{p}+\frac{1}{p+1}\right)z^2j_p(z)^2\,, \\
C_\ell(z) &= 2\left(\frac{\pi}{2}-A_\ell(z)+i\left(B_\ell(z)+\gamma+\ln(2z)\right)\right)-i\left(1+\sum_{p=1}^{\ell-1}\left(\frac{1}{p}+\frac{1}{p+1}\right)z(-1)^pR_{2p,\frac{1}{2}-p}(z)\right),
\end{aligned} \tag{A3}$$

for $\ell \geq 1$, and

$$
\begin{aligned}
A_0(z) &= \mathrm{Si}(2z)\,,\\
B_0(z) &= \mathrm{Ci}(2z) - \gamma - \ln(2z)\,,\\
C_0(z) &= 2\left(\frac{\pi}{2} - \mathrm{Si}(2z) + i\mathrm{Ci}(2z)\right),
\end{aligned} \tag{A4}
$$

for $\ell = 0$. It should be noted that the expressions (A1) and (A2) are valid only for $\omega \geq 0$. For $\omega < 0$ (i.e, for $z < 0$), they should be replaced by $\overline{X^{\rm in}_\ell(|z|)}$ and $\overline{X^{\rm up}_\ell(|z|)}$, respectively.

APPENDIX B: THE CHRZANOWSKI TRANSFORMATION

In this appendix, we summarize the Chrzanowski method [8] to construct the metric pertubation from the Weyl scalar and describe the modification of differential operators necessary to extract the tail part of the metric perturbation.

Using the method of Chrzanowski [8], we can generate metric perturbations in either the in-going radiation gauge or the out-going radiation gauge from the Weyl scalar. We first introduce the following differential operators on the Kerr spacetime:

$$
\begin{aligned}
\hat{h}^{\rm in}_{nn} &= -(\bar{\delta} + \alpha + 3\bar{\beta} - \bar{\tau})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau})\,,\\
\hat{h}^{\rm in}_{mm} &= -(D - \bar{\rho} + 3\bar{\epsilon} - \epsilon)(D + 3\bar{\rho} + 4\bar{\epsilon})\,,\\
\hat{h}^{\rm in}_{nm} &= -1/2\big\{(D + \rho - \bar{\rho} + \epsilon + 3\bar{\epsilon})(\bar{\delta} + 4\bar{\beta} + 3\bar{\tau})\\
&\qquad +(\bar{\delta} + 3\bar{\beta} - \alpha - \pi - \bar{\tau})(D + 3\bar{\rho} + 4\bar{\epsilon})\big\}\,,
\end{aligned} \tag{B1}
$$

and

$$
\begin{aligned}
\hat{h}^{\rm out}_{ll} &= -(\delta + \bar{\pi} - 3\bar{\alpha} - \beta)(\delta - 3\bar{\pi} - 4\bar{\alpha})(\bar{\rho})^{-4}\,,\\
\hat{h}^{\rm out}_{\bar{m}\bar{m}} &= -(\Delta + \gamma - 3\bar{\gamma} + \bar{\mu})(\Delta - 4\bar{\gamma} - 3\bar{\mu})(\bar{\rho})^{-4}\,,\\
\hat{h}^{\rm out}_{l\bar{m}} &= -1/2\big\{(\delta + \tau + \bar{\pi} - 3\bar{\alpha} + \beta)(\Delta - 4\bar{\gamma} - 3\bar{\mu})\\
&\qquad +(\Delta - \gamma - 3\bar{\gamma} + \bar{\mu} - \mu)(\delta - 3\bar{\pi} - 4\bar{\alpha})\big\}(\bar{\rho})^{-4}\,.
\end{aligned} \tag{B2}
$$

Here $D = l^\mu\nabla_\mu$, $\Delta = n^\mu\nabla_\mu$, $\delta = m^\mu\nabla_\mu$ and $\bar{\delta} = \bar{m}^\mu\nabla_\mu$, where $\{\ell^\mu,\, n^\mu,\, m^\mu,\, \bar{m}^\mu\}$ is the Kinnersley null tetrad, ρ, μ, τ, π, ϵ, γ, α and β are the spin coefficients of the Newman-Penrose formalism [30], and the overline indicates complex conjugation.

With these differential operators, we have the following metric perturbation in the Fourier-harmonic expansion in the ingoing and outgoing radiation gauges:

$$
h^{\rm in}_{\mu\nu}(\ell m\omega) = \big\{l_\mu l_\nu \hat{h}^{\rm in}_{nn} + \bar{m}_\mu\bar{m}_\nu\hat{h}^{\rm in}_{mm} - (l_\mu\bar{m}_\nu + \bar{m}_\mu l_\nu)\hat{h}^{\rm in}_{nm}\big\}{}_{-2}R_{\ell m\omega}(r){}_2Y_{\ell m}(\theta,\phi)e^{-i\omega t}\,, \tag{B3}
$$

$$
h^{\rm out}_{\mu\nu}(\ell m\omega) = \big\{n_\mu n_\nu \hat{h}^{\rm out}_{ll} + m_\mu m_\nu\hat{h}^{\rm out}_{\bar{m}\bar{m}} - (n_\mu m_\nu + m_\mu n_\nu)\hat{h}^{\rm out}_{l\bar{m}}\big\}{}_{2}R_{\ell m\omega}(r){}_{-2}Y_{\ell m}(\theta,\phi)e^{-i\omega t}\,. \tag{B4}
$$

Here ${}_sR_{lm\omega}(r)$ is the Teukolsky function of spin index s, and ${}_sY_{\ell m}(\theta,\phi)$ is the spin weighted spherical harmonic function.

Now we specify the background spacetime to be a Schwarzschild blackhole. We assume operands are expanded in the Fourier-harmonic functions. We use the pure radial and angular operators introduced by Chandrasekhar [10],

$$
\mathcal{D}_n = \partial_r - \frac{ir^2\omega}{\Delta_K} + 2n\frac{r-M}{\Delta_K}\,, \quad \mathcal{D}^\dagger_n = \partial_r + \frac{ir^2\omega}{\Delta_K} + 2n\frac{r-M}{\Delta_K}\,, \tag{B5}
$$

$$
\mathcal{L}_n = \partial_\theta + m\frac{1}{\sin\theta} + n\cot\theta\,, \quad \mathcal{L}^\dagger_n = \partial_\theta - m\frac{1}{\sin\theta} + n\cot\theta\,, \tag{B6}
$$

where $\Delta_K = r(r-2M)$. The Newman-Penrose operators and spin coefficients become

$$
D \,=\, \mathcal{D}_0\,, \quad \Delta \,=\, -\frac{\Delta_K}{2r^2}\mathcal{D}^\dagger_0\,, \quad \delta \,=\, \frac{1}{\sqrt{2}\,r}\mathcal{L}^\dagger_0\,, \tag{B7}
$$

$$
\rho \,=\, -\frac{1}{r}\,, \quad -\alpha \,=\, \beta \,=\, \frac{\cot\theta}{2\sqrt{2}\,r}\,, \quad \mu \,=\, -\frac{\Delta_K}{2r^3}\,, \quad \gamma \,=\, \frac{M}{2r^2}\,. \tag{B8}
$$

The radial differential operator $\mathcal{L}$ introduced in Eq. (2.16) is related to the operator D in the above as $\mathcal{L} = fD$, where $f = (r-2M)/r$. Then all the operators defined in Eqs. (B1) and (B2) are expressed as

$$\hat{h}^{\rm in}_{nn} = -\frac{1}{2r^2}\mathcal{L}_1\mathcal{L}_2\,, \quad \hat{h}^{\rm in}_{nm} = -\frac{1}{\sqrt{2}\,r}\Big(\mathcal{D}_0 - \frac{2}{r}\Big)\mathcal{L}_2\,, \quad \hat{h}^{\rm in}_{mm} = -\Big(\mathcal{D}_0 + \frac{1}{r}\Big)\Big(\mathcal{D}_0 - \frac{3}{r}\Big)\,, \tag{B9}$$

$$\hat{h}^{\rm out}_{ll} = -\frac{r^2}{2}\mathcal{L}_1^\dagger\mathcal{L}_2^\dagger\,, \quad \hat{h}^{\rm out}_{l\bar{m}} = \frac{r\Delta_K}{2\sqrt{2}}\Big(\mathcal{D}_2^\dagger - \frac{2}{r}\Big)\mathcal{L}_2^\dagger\,, \quad \hat{h}^{\rm out}_{\bar{m}\bar{m}} = -\frac{\Delta_K^2}{4}\Big(\mathcal{D}_2^\dagger + \frac{1}{r}\Big)\Big(\mathcal{D}_2^\dagger - \frac{3}{r}\Big)\,. \tag{B10}$$

In the remainder of this appendix, operators used to derive the metric perturbation from the Regge-Wheeler variable are given. We consider the metric perturbation in the ingoing radiation gauge.

First, note that, when operating on a quantity of spin weight s, the angular operators $\mathcal{L}_s$ and $\mathcal{L}^\dagger_{-s}$ are just the "edth" operators given in Eq. (2.3), except for the over-all sign:

$$\eth = -\mathcal{L}^\dagger_{-s}\,, \quad \bar{\eth} = -\mathcal{L}_s\,. \tag{B11}$$

Hence

$$\mathcal{L}_2\,({}_2Y_{\ell m}) = -\bar{\eth}\,({}_2Y_{\ell m}) = [(\ell-1)(\ell+2)]^{1/2}{}_1Y_{\ell m}\,,$$
$$\mathcal{L}_1\mathcal{L}_2\,({}_2Y_{\ell m}) = \bar{\eth}^2\,({}_2Y_{\ell m}) = [(\ell-1)\ell(\ell+1)(\ell+2)]^{1/2}{}_0Y_{\ell m}\,.$$

Accordingly, we put

$$\begin{aligned}\hat{h}^{\rm in}_{nn} &= {}_0p_\ell\,{}_0\hat{d}_\omega\,,\\ \hat{h}^{\rm in}_{nm} &= {}_{-1}p_\ell\,{}_1\hat{d}_\omega\,,\\ \hat{h}^{\rm in}_{mm} &= {}_{-2}p_\ell\,{}_2\hat{d}_\omega\,,\end{aligned} \tag{B12}$$

where the coefficients ${}_sp_\ell$ $(s=0,-1,-2)$ are defined by

$$\begin{aligned}{}_0p_\ell &= 2[(\ell-1)\ell(\ell+1)(\ell+2)]^{1/2}\,,\\ {}_{-1}p_\ell &= 2[2(\ell-1)(\ell+2)]^{1/2}\,,\\ {}_{-2}p_\ell &= 1\,,\end{aligned} \tag{B13}$$

and ${}_s\hat{d}_\omega$ $(s=0,1,2)$ are the radial differential operators defined by

$$\begin{aligned}{}_0\hat{d}_\omega &= -\frac{1}{4\,r^2}\,,\\ {}_1\hat{d}_\omega &= -\frac{1}{4}r\,f^{-1}\bar{\mathcal{L}}\,r^{-2}\,,\\ {}_2\hat{d}_\omega &= -r^{-1}f^{-1}\bar{\mathcal{L}}\,r^4 f^{-1}\bar{\mathcal{L}}\,r^{-3}\,.\end{aligned} \tag{B14}$$

Next, we rewrite the Green function for the Weyl scalar ψ_4 in a more convenient manner, so that the source term is divided into the tetrad components of the energy-momentum tensor that have specific spin weights, ${}_sT_{\omega\ell m}$, as defined by Eq. (D2). For this purpose, we introduce the operators ${}_sD_\omega$ and ${}_sD^\dagger_\omega$ defined in Poisson and Sasaki [20]:

$$\begin{aligned}{}_0D_\omega &= {}_0D^\dagger_\omega = r^4\,,\\ {}_{-1}D_\omega &= r^2 f\mathcal{L}\,r^3 f^{-1}\,,\\ {}_{-2}D_\omega &= rf\,\mathcal{L}\,r^4 f^{-1}\mathcal{L}\,r\,,\\ {}_{-1}D^\dagger_\omega &= -r^7\bar{\mathcal{L}}\,r^{-2}\,,\\ {}_{-2}D^\dagger_\omega &= r^5 f\,\bar{\mathcal{L}}\,r^4 f^{-1}\bar{\mathcal{L}}\,r^{-3}\,.\end{aligned} \tag{B15}$$

Using these operators, we can rewrite the expression for ψ_4 given by Eq. (4.2) as

$$\begin{aligned} r^4\psi_4(x) &= \int\frac{d\omega}{2\pi}\sum_{\ell m}\left[\int_{2M}^\infty dr'\,{}_{-2}\mathcal{G}_{\ell m\omega}(r,r')\,T_{\omega\ell m}(r')\,r'^{-2}(r'-2M)^{-2}\right]{}_{-2}Y_{\ell m}(\Omega)e^{-i\omega t}\\ &= \int\frac{d\omega}{2\pi}\sum_{\ell m}\left[\sum_s\int_{2M}^\infty dr' r'^{-2}(r'-2M)^{-2}{}_{(s)}G_{\ell m\omega}(r,r')\,{}_sT_{\omega\ell m}(r')\right]{}_{-2}Y_{\ell m}(\Omega)e^{-i\omega t}\,,\end{aligned} \tag{B16}$$

where the Green Functions ${}_{(s)}G_{\ell m\omega}(r,r')$ $(s=0,-1,-2)$ are defined by

$$
{}_{(s)}G_{\ell m\omega}(r,r') = {}_s p_\ell \, {}_s D^\dagger_\omega(r') \, {}_{-2}\mathcal{G}_{\ell m\omega}(r,r') \,, \tag{B17}
$$

with ${}_s D^\dagger_\omega(r')$ acting on the r'-dependent part of ${}_{-2}\mathcal{G}_{\ell m\omega}(r,r')$. The relation between the Teukolsky radial function ${}_{-2}R_{\ell m\omega}(r)$ to be used in the Chrzanowski transformation (B3) and the radial part of $r^4\psi_4(x)$ is

$$
{}_{-2}R_{\ell m\omega}(r) = \frac{32}{({}_0p_\ell)^2} \sum_s \int_{2M}^\infty dr' r'^{-2}(r'-2M)^{-2} {}_{(s)}G_{\ell m\omega}(r,r') {}_s T_{\omega\ell m}(r') \,. \tag{B18}
$$

Here, however, we wish to construct a Green function that directly gives the metric perturbation. For this purpose, we introduce the differential operators ${}_s\Delta_\omega = {}_s d_\omega \, C_\omega$ and ${}_s\Lambda_\omega = {}_s D^\dagger_\omega \, C_\omega$, where ${}_s d_\omega$ and ${}_s D^\dagger_\omega$ are defined by Eqs. (B14) and (B15), respectively, and C_ω is the Chandrasekhar operator defined by Eq. (2.15). Explicitly, ${}_s\Delta_\omega$ and ${}_s\Lambda_\omega$ are given by

$$
{}_0\Delta_\omega = -\frac{1}{4} f \, \mathcal{L} \, f^{-1} \mathcal{L} \, r = -\frac{1}{4} r^{-1} \, {}_0\Gamma_\omega \,, \tag{B19}
$$

$$
{}_1\Delta_\omega = -\frac{1}{4} \, r \, f^{-1} \bar{\mathcal{L}} \, f \, \mathcal{L} \, f^{-1} \mathcal{L} \, r = \frac{1}{4} r^{-1} f^{-1} \, {}_{-1}\Gamma_\omega \,, \tag{B20}
$$

$$
{}_2\Delta_\omega = -r^{-1} f^{-1} \bar{\mathcal{L}} \, r^4 f^{-1} \bar{\mathcal{L}} \, r^{-1} f \, \mathcal{L} f^{-1} \mathcal{L} \, r = -r^{-1} f^{-2} \, {}_{-2}\Gamma_\omega \,, \tag{B21}
$$

where the operators ${}_s\Gamma_\omega$ are introduced in [20] and are given by

$$
\begin{aligned}
{}_0\Gamma_\omega &= (\omega\, r)^{-1} C_\omega \\
&= 2\left(1 - \frac{3\,M}{r} + i\,\omega r\right) r\,f \frac{d}{dr} + f\left(\ell(\ell+1) - \frac{6\,M}{r}\right) + 2\,i\,\omega r \left(1 - \frac{3\,M}{r} + i\,\omega r\right) \,,
\end{aligned} \tag{B22}
$$

$$
\begin{aligned}
{}_{-1}\Gamma_\omega &= -r^2 \bar{\mathcal{L}} \, f \, \mathcal{L} \, f^{-1} \mathcal{L} \, r \\
&= -f\left[(\ell(\ell+1) + 2\,i\,\omega r) \, r\,f \frac{d}{dr} + \ell(\ell+1)(f + i\,\omega r) - 2(\omega r)^2 \right] \,,
\end{aligned} \tag{B23}
$$

$$
\begin{aligned}
{}_{-2}\Gamma_\omega &= f \, \bar{\mathcal{L}} \, r^4 f^{-1} \bar{\mathcal{L}} r^{-1} f \, \mathcal{L} \, f^{-1} \mathcal{L} \, r \\
&= f^2 \left[2\left((\ell-1)(\ell+2) + \frac{6\,M}{r} \right) r\,f \frac{d}{dr} + (\ell-1)(\ell+2)(\ell(\ell+1) + 2\,i\,\omega r) - 12 \frac{f\,M}{r} \right] \,.
\end{aligned} \tag{B24}
$$

With these newly introduced operators, the regularized Green functions for the components of the metric perturbation can be written as

$$
G^{\ell m\omega}_{(s,s')}(r,r') = -\frac{64\,\pi}{({}_0p_\ell)^2} \frac{W(X^{\rm up}_\ell, X^{\rm in}_\ell)}{W(R^{\rm up}_\ell, R^{\rm in}_\ell)} \, \chi^{\rm in}_\ell \, \chi^{\rm up}_\ell \, {}_s p_\ell \, {}_{s'}p_\ell \, {}_s\Delta_\omega(r) \, {}_{s'}\Lambda_\omega(r') \, {}_{-2}G^{\rm tail}_{\ell m\omega}(r,r') \,, \tag{B25}
$$

where the coefficeints $\chi^{\rm in}_\ell$ and $\chi^{\rm up}_\ell$ are defined in Eq. (2.17). Note that the operators ${}_s\Delta_\omega$ and ${}_s\Gamma_\omega$ defined above become the modified differential operators by reducing the higher r-derivatives using the homogeneous Regge-Wheeler equation. As discussed in Sec. IV, these operators transform the tail of the Regge-Wheeler Green function into the tail of the Green functions for the metric components. If the original higher differential operators are used, the direct part is partially induced, as shown in Appendix C.

APPENDIX C: THE INDUCED DIRECT PART OF THE GREEN FUNCTION

In Appendix B, we introduced modified differential operators that differ from the original Chandrasekhar and Chrzanowski operators in order to obtain the tail part of the metric perturbation. This is because the direct part is partially induced by taking derivatives of the tail part of the Regge-Wheeler Green function, and such an induced part must be subtracted out. In this appendix, we explicitly show this phenomenon at 1PN order. Then we illustratively summarize our prescription to subtract the induced direct part from the Green function for the metric perturbation.

The homogeneous radial Regge-Wheeler function satisfies

$$
\left[\mathcal{O}_0 + \epsilon \left(\mathcal{O}_1 + \frac{s^2}{z^3} \right) \right] X_\ell = 0 \,, \tag{C1}
$$

where $z = \omega r$, $\epsilon = 2\omega M$, and the operators $\mathcal{O}_0$ and $\mathcal{O}_1$ are given by

$$\begin{aligned}\mathcal{O}_0 &= \frac{d^2}{dz^2} + 1 - \frac{\ell(\ell+1)}{z^2}\,,\\ \mathcal{O}_1 &= \frac{2}{z}\frac{d^2}{dz^2} + \frac{1}{z^2}\frac{d}{dz} + \frac{\ell(\ell+1)}{z^3} - \frac{1}{z^3}\,.\end{aligned} \tag{C2}$$

The Green function is constructed from the homogeneous functions as $X_\ell^{\rm in}(z)X_\ell^{\rm up}(z')$. Therefore we consider this form. The homogeneous Regge-Wheeler functions can be expanded to 1PM order as

$$X_\ell = X_0 + \epsilon\left(X_1^d + s^2 X_1^t\right) + O(\epsilon^2)\,, \tag{C3}$$

where X_1^d and X_1^t are the parts of the 1PM correction independent of s and dependent on s, respectively. When we further adopt the post-Newtonian expansion, they correspond to the parts of the homogeneous function that contribute to the direct and tail parts of the Green function, respectively, as discussed in Sec. III. Specifically, the direct part of the Green function consists of $X_0(z)X_0(z')$ and $X_0(z)X_1^d(z') + X_1^d(z)X_0(z')$, and the tail part consists of $X_0(z)X_1^t(z') + X_1^t(z)X_0(z')$.

We substitute Eq. (C3) into Eq. (C1) and collect terms of the same order in ϵ. Then we find

$$\begin{aligned}&\mathcal{O}_0\, X_0 = 0\,,\\ &\mathcal{O}_1\, X_0 + \mathcal{O}_0\, X_1^d + s^2\left(\mathcal{O}_0\, X_1^t + \frac{1}{z^3}\, X_0\right) = 0\,.\end{aligned} \tag{C4}$$

Since X_0 and X_1^d are independent of s, it follows that

$$\begin{aligned}&\mathcal{O}_1\, X_0 + \mathcal{O}_0\, X_1^d = 0\,,\\ &\mathcal{O}_0\, X_1^t + \frac{1}{z^3}\, X_0 = 0\,.\end{aligned} \tag{C5}$$

We find from the second equation of the above that the tail part of the Regge-Wheeler Green function is transformed into the direct part by the operation $\mathcal{O}_0$.

To subtract out such an induced direct part, it is therefore nessesary to reduce the second-order derivative by the equation $\mathcal{O}_0 X_1^t = 0$ whenever second- or higher-order derivatives operate on the Regge-Wheeler Green function. Although we do not have a general proof, we expect the same to be true at any order of the post-Newtonian expansion. Thus, in general, the prescription is to reduce the higher-order derivatives on the tail part by the equation

$$\left[\mathcal{O}_0 + \epsilon\left(\mathcal{O}_1 + s^2\,\frac{1}{z^3}\right)\right]\, X_\ell^t = 0\,, \tag{C6}$$

where X_ℓ^t is the part of the homogeneous Regge-Wheeler funtion that constitutes the tail part of the Green function.

It may be useful to summarize the above prescription illustratively. First, one constructs the full Regge-Wheeler Green function, say G_{RW}. Then the full Green function for the metric perturbation, say G, is obtained as

$$G(x,x') = \mathcal{O}(x)\mathcal{O}'(x')G_{\rm RW}(x,x'), \tag{C7}$$

where $\mathcal{O}$ and $\mathcal{O}'$ are some higher-order differential operators. As long as $x \neq x'$, one can then reduce the higher-order derivatives in $\mathcal{O}$ and $\mathcal{O}'$ by using the homogeneous Regge-Wheeler equation without any problem. Let us denote the reduced operators thus obtained by $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}'$. Then we have

$$G(x,x') = \tilde{\mathcal{O}}(x)\tilde{\mathcal{O}}'(x')G_{\rm RW}(x,x'). \tag{C8}$$

By construction, the operators $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}'$ contain only the first derivatives. Now we divide $G_{\rm RW}$ into the direct part $G_{\rm RW}^{\rm dir}$ and the tail part $G_{\rm RW}^{\rm tail}$. Then the tail part of the Green function for the metric is given by

$$G^{\rm tail}(x,x') = \tilde{\mathcal{O}}(x)\tilde{\mathcal{O}}'(x')G_{\rm RW}^{\rm tail}(x,x'). \tag{C9}$$

This is our prescription to obtain the tail part of the Green function for the metric perturbation.

APPENDIX D: NOTATION FOR THE SOURCE TERM

The source term of the inhomogeneous Teukolsky equation is constructed from the energy-momentum tensor of a point particle:

$$T^{\alpha\beta}(x) = \mu \int d\tau u^\alpha u^\beta \delta^{(4)}[x - x'(\tau)] \,. \tag{D1}$$

The first step is to obtain the projections by the null tetrad, $T_{\alpha\beta}\, n^\alpha\, n^\beta$, $T_{\alpha\beta}\, n^\alpha\, \bar{m}^\beta$ and $T_{\alpha\beta}\, \bar{m}^\alpha\, \bar{m}^\beta$. Then we calculate the Fourier-harmonic components according to

$$_sT_{\omega\ell m} = \frac{1}{2\pi} \int dt\, d\Omega\, {}_sT(x)\, \overline{{}_sY_{\ell m}(\Omega)}\, e^{i\omega t} \,, \tag{D2}$$

where the quantities $_sT$ $(s = 0, -1, -2)$ are defined by

$${}_0T = T_{\alpha\beta}\, n^\alpha\, n^\beta \,, \quad {}_{-1}T = T_{\alpha\beta}\, n^\alpha\, \bar{m}^\beta \,, \quad {}_{-2}T = T_{\alpha\beta}\, \bar{m}^\alpha\, \bar{m}^\beta \,. \tag{D3}$$

Then the source term $T_{\omega\ell m}$ for the radial Teukolsky equation is obtained by operating on $_sT_{\omega\ell m}$ with relevant diferential operators and then by summing over s as

$$T_{\omega\ell m} = \sum_s {}_sp_\ell\, {}_sD_\omega\, {}_sT_{\omega\ell m} \,, \tag{D4}$$

where the $_sD_\omega$ are the differential operators defined in Eq. (B15).

We note that the source $_{-2}\mathcal{T}$ of the Teukolsky equation (2.1) is related to the above $T_{\omega\ell m}$ as

$${}_{-2}\mathcal{T}_{\omega\ell m} = \frac{1}{2\pi} \int dt\, d\Omega\, {}_{-2}\mathcal{T}(x)\, \overline{{}_{-2}Y_{\ell m}(\Omega)}\, e^{i\omega t} = \frac{T_{\omega\ell m}}{2\, r^2} \,. \tag{D5}$$

In this paper, we use the notation of Poisson and Sasaki [20] for the source term and use $T_{\omega\ell m}$.

Note that Chrzanowski [8] uses different notation for the source term T in the Teukolsky equation. Denoting Chrzanowski's T by $T_{\rm C}$, it is related to $T_{\omega\ell m}$ and $_{-2}\mathcal{T}_{\omega\ell m}$ as

$$T_{\rm C} = -2\,\pi\, T_{\omega\ell m} = -4\,\pi\, r^2\, {}_{-2}\mathcal{T}_{\omega\ell m} \,. \tag{D6}$$

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[1] For a review, see e.g., Y. Mino, M. Sasaki, M. Shibata, T. Tagoshi and T. Tanaka, Prog. Theor. Phys. Suppl. **128**, 1 (1997).
[2] B. Carter, Phys. Rev. **174**, 1559 (1968).
[3] Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. **D55**, 3457 (1997).
[4] T. C. Quin and R. M. Wald, Phys. Rev. **D56**, 3381 (1997).
[5] Y. Mino, Prog. Theor. Phys. **99**, 79 (1998).
[6] A. Teukolsky, Astrophys. J. **185**, 635 (1973).
[7] S. Mano, H. Suzuki and E. Takasugi, Prog. Theor. Phys. **95**, 1079 (1996).
[8] P. L. Chrzanowski, Phys. Rev. **D11**, 2042 (1975).
[9] Y. Mino and H. Nakano, Prog. Theor. Phys. **100**, 507 (1998).
[10] S. Chandrasekhar, Proc. R. Soc. (London) **A343**, 289 (1975); *Mathematical Theory of Black Holes* (Oxford University Press, 1983).
[11] M. Sasaki and T. Nakamura, Phys. Lett. **A89**, 68 (1982); Prog. Theor. Phys. **67**, 1788 (1982).
[12] L. Barack and A. Ori, Phys. Rev. **D61**, 061502 (2000).
[13] L. M. Burko, Class. Quant. Grav. **17**, 227 (2000).
[14] L. M. Burko, Phys. Rev. Lett. **84**, 4529 (2000).
[15] L. Barack and L. M. Burko, Phys. Rev. **D62**, 084040 (2000).
[16] L. Barack, Phys. Rev. **D62**, 084027 (2000).
[17] C. O. Lousto, Phys. Rev. Lett. **84**, 5251 (2000).
[18] L. M. Burko, Y. -T. Liu and Y. Soen, Phys. Rev. **D63**, 024015 (2001).

[19] P. L. Chrzanowski and C. W. Misner, Phys. Rev. **D10**, 1701 (1974).
[20] E. Poisson and M. Sasaki, Phys. Rev. **D51**, 5753 (1994).
[21] S. W. Leonard and E. Poisson, Phys. Rev. **D56**, 4789 (1997).
[22] B. S. DeWitt and R. W. Brehme, Ann. of Phys. **9**, 220 (1960).
[23] K. S. Thorne and S. J. Kovacs, Astrophys. J. **200**, 245 (1975).
[24] F. Zerilli, Phys. Rev. **D2**, 2141 (1970).
[25] L. Blanchet, B. R. Iyer, C. M. Will and A. G. Wiseman, Class. Quant. Grav. **13**, 575 (1996).
[26] C. M. Will and A. G. Wiseman, Phys. Rev. **D54**, 4813 (1996).
[27] L. Blanchet, G. Faye and B. Ponsot, Phys. Rev. **D58**, 124002 (1998).
[28] L. Blanchet and G. Faye, gr-qc/0007051, submitted to Phys. Rev. **D**.
[29] H. Nakano and M. Sasaki, in preparation.
[30] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).